Inverse method for the investigation of nonlinear instabilities associated with negative-energy modes

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Earlier work on explosively unstable similarity solutions of Hamilton's equations with Hamiltonians homogeneous of degree N and satisfying resonance conditions is applied to study the nonlinear stability of linearly stable equilibria with neighboring positive- and negative-energy waves. A multiple-time-scale expansion near equilibrium yields a Hamiltonian system of the assumed structure. In the inverse method an explosively unstable similarity solution is assumed and one solves for the coefficients of the terms in a Hamiltonian of some given structure. Through some general arguments and many examples one concludes that explosively unstable solutions occur generally for wide ranges of coefficient values. Hence the original equilibrium is nonlinearly unstable for wide ranges of interaction parameters.

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I. INTRODUCTION

In 1925 Cherry [1] discussed two oscillators of positive and negative energy that are nonlinearly coupled in a special way, and presented a class of exact solutions of the nonlinear equations showing explosive instability independent of the strength of the nonlinearity and the initial amplitudes, although linearized theory predicts absolute stability. (For references on nonlinear instabilities see J. Weiland and H. Wilhelmsson [2] and H. Wilhelmsson [3]; see also Ref. [4].) Pfirsch [5] has reformulated Cherry's example and generalized it to three oscillators satisfying the resonance condition $\sum_i \omega_i = 0$. If this condition is not satisfied, the system is still explosively unstable, but the initial perturbations must exceed a certain threshold. It is easy to generalize this further to an arbitrary number of oscillators, but the coupling terms are restricted to be Cherry-like. In the quantum mechanical language of Refs. [5] and [6], this means that the coupling terms consist of products of creation operators only and annihilation operators only. Such coupling terms do not represent the general case, in which a sum of mixed products of annihilation and creation operators occurs in the coupling terms.

This paper explores the occurrence of explosive instabilities in dynamical systems in the neighborhood of a linearly stable equilibrium point where the frequencies of the linearized motion satisfy resonance conditions. The multiple time scale formalism has been applied previously to this problem [6], and this paper employs many of the results developed there. The occurrence of resonance introduces an additional constant of the motion [6], which ensures that for such systems nonlinear instabilities may occur only if negative-energy waves are present. The close connection between nonlinear instability and the existence of negative-energy waves has been considered by many authors [7–9], and has led to extensive literature in plasma physics examining the conditions for the existence of negative-energy waves [10–12], and references cited therein. Related dynamical problems also appear in the context of nonlinear optics [13,14]. This paper examines the connection between negative-energy waves and nonlinear instabilities.

If the origin is a linearly stable point of equilibrium for a dynamical system with real-valued Hamiltonian $\tilde{H}(\zeta_j,\zeta_k^*)$, $j,k=1,2,\ldots,M$, then one may assume that

$$\tilde{H}=\sum_{l=1}^{n}\omega_l\zeta_l^*\zeta_l+V(\zeta_j,\zeta_k^*)\ , \tag{1}$$

where $V(\zeta_j,\zeta_k^*)$ has many derivatives with respect to its arguments and the function and all its first and second derivatives vanish at the origin and ω_l are real constants. In order to examine solutions near the point of equilibrium, one may set

$$\zeta_j=\epsilon\eta_j\ ,\quad j=1,2,\ldots,M\ , \tag{2}$$

where ϵ is small, so that the variational form associated with Hamiltonian's equations becomes

$$i\sum_{l=1}^{M}\zeta_l^*d\zeta_l-\tilde{H}dt=\epsilon^2\left[i\sum_{l=1}^{M}\eta_l^*d\eta_l-H(\eta_j,\eta_k^*)dt\right]\ , \tag{3}$$

where

$$H(\eta_j,\eta_k^*)=\sum_{l=1}^{M}\omega_l\eta_l\eta_l^*+\epsilon^{-2}V(\epsilon\eta_j,\epsilon\eta_k^*)\ . \tag{4}$$

The interaction function $\epsilon^{-2}V(\epsilon\eta_j,\epsilon\eta_k^*)$ is a sum of homogeneous polynomials of degree N in η_j and η_k^* multiplied by ϵ^{N-2}, where $N\geq 3$, plus an error term which is multiplied by a higher power of ϵ. The equations of motion obtained from (3) and (4) are

$$i\dot{\eta}_l=\omega_l\eta_l+\epsilon\frac{\partial}{\partial\eta_l^*}\epsilon^{-3}V(\epsilon\eta_j,\epsilon\eta_k^*) , \tag{5}$$

where the second term on the right hand side of (5) is small in ϵ. Hamilton's equations in the form (5) make explicit the linear stability of the original system (1), in that for $\epsilon=0$, corresponding to small amplitude solutions, c.f. (2), the solutions are purely harmonic oscillators. The question then naturally arises: Do the small, nonlinear terms in (5) cause the solutions to become unstable? That is, do the small nonlinear terms in (5) cause the solution to move away from the origin?

The system (5) has been treated in [6] with the multiple time scale formalism, and if one introduces $\xi_l(t)$ by

$$\eta_l(t)=\xi_l(t)\epsilon^{-i\omega_l t} . \tag{6}$$

then in lowest nontrivial order $\xi_l(t)$ satisfies

$$i\dot{\xi}_l\equiv i\frac{d\xi_l}{d\tau}=\frac{\partial W}{\partial\xi_l^*}(\xi_j,\xi_k^*) , \quad l=1,2,\ldots,M, \tag{7}$$

where

$$W(\xi_j,\xi_k^*)=\lim_{t\to\infty}\frac{1}{t}\int_0^t dt' V_N(\epsilon^{-i\omega_j t'}\xi_j,\epsilon^{i\omega_k t'}\xi_k^*) , \tag{8}$$

$V_N(\eta_j,\eta_k^*)$ is the lowest order homogeneous polynomial of degree N from $V(\xi_j,\xi_k^*)$ which generates a nonvanishing $W(\xi_j,\xi_k^*)$, and

$$\tau=\epsilon^{N-2}t . \tag{9}$$

The assumption that $W(\xi_j,\xi_k^*)$, a real-valued homogeneous polynomial of degree N, is nonzero has as its consequence that the linearized motion possesses resonances, that is, there are positive integers n_j such that

$$\sum_{j=1}^{M} n_j\omega_j=0 . \tag{10}$$

More precisely, for the monomial

$$\prod_{j=1}^{M}\xi_j^{m_j}(\xi_j^*)^{n_j} \tag{11}$$

to appear in W, m_j, n_j, and ω_j must satisfy

$$\sum_{j=1}^{M}(m_j+n_j)=N \tag{12}$$

and

$$\sum_{j=1}^{M}(m_j-n_j)\omega_j=0 . \tag{13}$$

Thus $W(\xi_j,\xi_k^*)$ is a real-valued sum of monomials of the form (11)–(13) multiplied by complex coefficients. If the system (7), an approximation to the original system (5), has solutions that move away from the origin, for instance, explosively unstable solutions, then it is reasonable to conclude that some solutions of the original system also move far from the original, although they might not be explosively unstable.

The following results are taken over directly from [6], where it was shown that $W(\xi_j,\xi_k^*)$ satisfies the transformation property:

$$W(e^{(\alpha+i\omega_j)\phi}\xi_j,e^{(\alpha-i\omega_k)\phi}\xi_k^*)=e^{N\alpha\phi}W(\xi_j,\xi_k^*) \tag{14}$$

[compare (11), (12), and (13)] and that any solution of (7) also satisfies

$$\sum_{j=1}^{M}\omega_j|\xi_j|^2=\text{const} , \tag{15}$$

as well as the standard relation for time-independent Hamiltonian systems

$$W(\xi_j,\xi_k^*)=\text{const} . \tag{16}$$

From the transformation property (14), it follows that (7) may possess explosive instability similarity solutions of the form

$$\xi_j(t)=\frac{\mu_j}{[1-(N-2)\alpha\tau]^{\alpha+i\omega_j\beta/[\alpha(N-2)]}} , \quad j=1,2,\ldots,M , \tag{17}$$

where α and β are real constants and μ_j are complex constants. In order that (17) be a solution of (7) the constants must satisfy

$$i(\alpha+i\omega_j\beta)\mu_j\equiv D_j\mu_j=\frac{\partial W}{\partial\mu_j^*}(\mu_k,\mu_l^*) ,$$
$$j=1,2,\ldots,M . \tag{18}$$

A further useful property of the similarity solutions (17) is that if μ_j, α, and β characterize a similarity solution, then also $\mu_j\epsilon^{A+i\omega_j B}$, $\alpha e^{-(N-2)A}$, $\beta e^{-(N-2)A}$, with e^A and B real, describe another similarity solution with τ displaced by a given amount. Thus, of the $2M+2$ real constants that characterize a similarity solution, $\text{Re}\mu_j$, $\text{Im}\mu_j$, α, and β, two may be specified independently of the interaction $W(\xi_j,\xi_k^*)$. This paper explores the realizability of the explosively unstable solution (17) for the dynamical systems under consideration.

In order that (17) represents an explosively unstable motion it is necessary that $\alpha\neq0$, which is assumed in the remainder of this paper. In the limiting case $\alpha\to0$, $\beta\neq0$ (17) reduces to

$$\xi_j(\tau)=\mu_j e^{i\omega_j\beta\tau} , \tag{19}$$

a solution not relevant to the study of explosive instabilities. Provided $\alpha\neq0$, it is clear that the constants on the right hand sides of (15) and (16) must vanish, so that

$$\sum_{j=1}^{M}\omega_j|\mu_j|^2=0 \tag{20}$$

and

$$W(\mu_j,\mu_k^*)=0 . \tag{21}$$

Both (12) and (13) may be obtained directly from (18) by the formation of linear combinations of the individual equations in (18), cf. [6].

Almost all the material in this paper relates to resonant interactions, nonetheless for the purposes of comparison it is useful to consider the Hamiltonian system (7) where $W(\xi_j,\xi_k^*)$ is a homogeneous polynomial of degree N, but the additional resonance conditions (11), (12), and (13) are not imposed. This case is referred to as nonresonant, and is a purely formal example; it cannot be obtained by the multiple time scale averaging employed in (8). Nonetheless the Hamiltonian $W(\xi_j,\xi_k^*)$ satisfies the transformation property (15) provided one sets $\omega_j \equiv 0, j=1,2,\ldots,M$. Thus, (17) is still a possible solution of the system (7) provided (18) holds and one sets $\omega_j \equiv 0$ in (17) and (18). This solution is characterized by $2M+1$ real constants, of which one may be specified independently of $W(\xi_j,\xi_k^*)$. Finally the identity (20) is trivial, while (21) is not.

II. THE INVERSE PROBLEM

Normally, one gives the interaction Hamiltonian $W(\xi_j,\xi_k^*)$, a homogeneous polynomial of degree N satisfying the transformation property (14) and one attempts to determine the coefficients characterizing the explosively unstable similarity solution μ_j, α, and β. The resulting problem, the determination of the solvability of the large, nonlinear algebraic system (18) is extremely difficult and typically leads to the establishment of conditions for the existence of real and positive solutions of a high order algebraic equation [5,6]. The inverse problem is to consider the quantities μ_j, α, and β as given, and to determine the coefficients in the Nth order polynomial characterizing the interaction Hamiltonian. The inverse problem is greatly simplified by its linear character. By its very nature the inverse problem changes the question from "does a particular Hamiltonian admit explosively unstable solutions?" to "for a Hamiltonian of a particular structure how large is the range of values of the coefficients in the Hamiltonian for which explosive instability solutions occur?"

The interaction Hamiltonian is composed of a sum of monomials of the form (11)–(13) multiplied by interaction constants γ, δ, ϵ, ζ, η, ρ, σ, etc. For the term $\gamma|\xi_j|^{2K}|\xi_k|^{2L}$ to be real, it is clearly necessary that γ be real. Thus, not all interaction constants may be explicitly complex. Here, it is assumed that Q real interaction constants characterize $W(\xi_j,\xi_k^*)$, and the inverse problem is to solve the real $2M$ equations given by the real and imaginary parts of (18) for the Q interaction constants. This problem is the standard problem of linear algebra and the solution is immediate. The coefficients of the interaction constants on the right hand side of (18) form a matrix of $2M$ rows and Q columns. The rank of the matrix can be no larger than $2M$, or Q, so that in any case there must be an integer $d \geq 0$ such that

$$Q \geq 2M - d \geq 0 \tag{22}$$

and that $2M-d$ is the rank of the matrix. In order that (18) be solvable the left hand side of (18) must satisfy d real consistency conditions, and these conditions involve μ_j, α, and β only. If these d consistency conditions admit nontrivial solutions then the system is solvable for the interaction constants, which depend on $Q-(2M-d)$ arbitrary parameters.

The linearity of the system also implies that if α, β, μ_j are a solution for one set of interaction constants, then $\lambda\alpha$, $\lambda\beta$, μ_j are a solution for the set of interaction constants of the first case each multiplied by λ. Thus, in the space of interaction constants, explosive instabilities occur on full lines passing through the origin. It suffices, then, to determine these points on the unit sphere in interaction constant space for which explosive instabilities exist. One may then speak of the fraction of the area of the sphere of explosive instability Hamiltonians. This fraction also represents the fraction of the entire space for which explosive instabilities occur.

For the case of resonant interactions $d \geq 1$, since (20) exhibits one dependency relation derivable from (18) no matter whether $Q \geq 2M$ or $Q < 2M$. Other dependency relations among the μ_j, α, and β may arise from constants of the motion of the system, or from the requirements imposed by the linear algebra problem and which have no clear dynamical origin. In the resonant case, assumed for the rest of this paragraph, the constraint clearly shows that explosive instabilities are possible only if both positive-energy waves, $\omega_j > 0$, and negative-energy waves, $\omega_j < 0$, are present. Otherwise only the trivial solutions $\mu_j = 0$ or stable solutions $\alpha = 0$ are possible. Under the assumption that the d real constraints admit nontrivial solutions, it is interesting to carry out a "counting argument" to try to predict the dimensionality of the set in interaction constant space in which explosive instabilities appear. If the dimensionality of the set equals the dimensionality of the full space, then by the previous arguments some nonzero fraction of the space of interaction constants is covered by explosive instabilities. The $2M+2$ real constants $\mathrm{Re}\mu_j$, $\mathrm{Im}\mu_j$, α, and β are constrained by d real relations; furthermore, two of the parameters may be chosen arbitrarily without affecting W. Thus, the linear system for the interaction constants depends on $2M+2-d-2=2M-d$ free parameters. The solution of the linear system for the Q interaction constants also depends on $Q-(2M-d)$ other free parameters. Hence the Q interaction constants are characterized by $Q-(2M-d)+2M-d=Q$ free constants. This counting argument would suggest that the dimensionality of the set of interactive constants whose Hamiltonians admit explosive instability solutions is Q. Of course, this argument is no proof, as the Jacobian of the transformation from interaction constants to free parameters may vanish identically. Nonetheless, it is very suggestive. This argument also assumes that d, an integer, is constant. It is clear that d may change its value on lower dimensional manifolds in the space of coefficients μ_j. A further, detailed discussion of additional pathologies appears of limited benefit. Nonetheless, in general one expects the set of solutions for the interaction constants to lie in a Q dimensional set. Correspondingly, a positive fraction of the set is associated with explosive instabilities.

The situation with a nonresonant Hamiltonian is not substantially different, other than $d=0$ is possible and that the explosive instability solution depends on only $2M+1$ parameters, since $\beta \equiv 0$. With appropriate modifications the counting argument leads to the same conclusion; the Q interaction constants depend on Q free parameters. After the examination of a number of explicit examples, the conclusions section summarizes the results.

III. EXAMPLES

In each of the following examples the frequencies of the interacting waves are chosen to be $\omega_n = n_j$, where the integers may be positive or negative. The choice of this special form of the frequencies allows realization of various resonance situations in a simple manner; it does not imply any significant restriction. The resonance condition (10) requires that the interacting modes separate into subsets, each of which has commensurable frequencies. The case we have treated, with only one set of commensurable frequencies, is, in fact, less restrictive than the cases with two or more sets of commensurable frequencies. In the latter case, the interaction Hamiltonians are even more restricted.

A. Third order interaction of four waves

(1) A "typical" case, $\omega_{-2} = -2$, $\omega_1 = 1$, $\omega_3 = 3$, $\omega_4 = 4$,

$$W = \gamma\xi_4(\xi_{-2})^2 + \gamma^*\xi_4^*(\xi_{-2}^*)^2 + \rho\xi_4\xi_1^*\xi_3^* + \rho^*\xi_4^*\xi_1\xi_3 + \sigma\xi_3\xi_1^*\xi_{-2} + \sigma^*\xi_3^*\xi_1\xi_{-2}^* + \delta\xi_1^2\xi_{-2} + \delta^*(\xi_1^*)^2\xi_{-2}^* \, . \tag{23}$$

The system (18) is

$$\begin{aligned} D_4\mu_4 &= \gamma^*(\mu_{-2}^*)^2 + \rho^*\mu_1\mu_3 \, , \\ D_3\mu_3 &= +\rho\mu_1^*\mu_4 + \sigma^*\mu_{-2}^*\mu_1 \, , \\ D_1\mu_1 &= +\rho\mu_3^*\mu_4 + \sigma\mu_{-2}\mu_3 + 2\delta^*\mu_{-2}\mu_1^* \, , \\ D_{-2}\mu_{-2} &= 2\gamma^*\mu_{-2}^*\mu_4^* + \sigma^*\mu_1\mu_3^* + \delta^*(\mu_1^*)^2 \, . \end{aligned} \tag{24}$$

Clearly (24) represents eight real equations in the eight unknowns, the real and imaginary parts of γ, δ, ρ, and σ. There exists at least one linear dependency relation,

$$-2|\mu_{-2}|^2 + |\mu_1|^2 + 3|\mu_3|^2 + 4|\mu_4|^2 = 0 \, , \tag{25}$$

and another linear combination of the relations shows that

$$W(\mu_j, \mu_k^*) = 0 \, . \tag{26}$$

It is convenient to replace the last complex equation of (24) with the two real relations (25) and (26). One can then solve the first three equations of (24) for γ, σ, and δ as functions of ρ and ρ^*, insert the solutions into (26), and one finds

$$-\beta(|\mu_1|^2 + 3|\mu_3|^2 + 8|\mu_4|^2) = \rho\mu_1^*\mu_3^*\mu_4 + \rho^*\mu_1\mu_3\mu_4 \, . \tag{27}$$

Thus, provided $\mu_1\mu_3\mu_4 \neq 0$, (27) determines $\mathrm{Re}(\rho\mu_1^*\mu_3^*\mu_4)$ while $\mathrm{Im}(\rho\mu_1^*\mu_3^*\mu_4)$ is arbitrary. The one arbitrary constant $\mathrm{Im}(\rho\mu_1^*\mu_3^*\mu_4)$ then completely determines γ, ρ, σ, and δ, provided that (25) holds. This is clearly the general situation of the inverse problem for which $d = 1$. One can also solve the direct problem, and with a little effort show that the domain of explosive instabilities in parameter space is eight dimensional.

(2) An "untypical" case, $\omega_{-2} = -2$, $\omega_{-1} = -1$, $\omega_1 = 1$, $\omega_2 = 2$,

$$W = \gamma\xi_{-2}\xi_1^2 + \gamma^*\xi_{-2}^*(\xi_1^*)^2 + \delta\xi_{-2}(\xi_{-1}^*)^2 + \delta^*\xi_{-2}^*(\xi_{-1})^2 + \epsilon\xi_2^*\xi_1^2 + \epsilon^*\xi_2(\xi_1^*)^2 + \zeta^*\xi_2(\xi_{-1})^2 + \zeta\xi_2^*(\xi_{-1}^*)^2 \, , \tag{28}$$

for which the linear system is

$$\begin{aligned} D_{-2}\mu_{-2} &= \gamma^*(\mu_1^*)^2 + \delta^*(\mu_{-1})^2 \, , \\ D_{-1}\mu_{-1} &= +2\delta\mu_{-2}\mu_{-1}^* + 2\zeta\mu_{-1}^*\mu_2^* \, , \\ D_1\mu_1 &= 2\gamma^*\mu_{-2}^*\mu_1^* + 2\epsilon^*\mu_1^*\mu_2 \\ D_2\mu_2 &= +\epsilon\mu_1^2 + \zeta(\mu_{-1}^*)^2 \, . \end{aligned} \tag{29}$$

The system (29) has great similarity to the previous case involving eight real equations and eight unknowns, the real and imaginary parts of γ, δ, ϵ, and ζ. The system is significantly different, however. If one takes the complex conjugate of the second and fourth relations, one sees that (29) is a complex linear system for σ^*, δ^*, ϵ^*, and ζ^*, so that one need not split (29) into real and imaginary parts, and moreover that

$$2D_{-2}|\mu_{-2}|^2 - D_{-1}^*|\mu_{-1}|^2 - D_1|\mu_1|^2 + 2D_2^*|\mu_2|^2 = 0 \tag{30}$$

so that one complex consistency relation exists. The imaginary part of (30), provided $\alpha \neq 0$, yields the known constraint

$$2|\mu_{-2}|^2 + |\mu_{-1}|^2 - |\mu_1|^2 - 2|\mu_2|^2 = 0 \tag{31}$$

while the real part, with (31), yields

$$\beta(|\mu_{-1}|^2 - |\mu_1|^2) = 0 \, . \tag{32}$$

For the dynamical system with Hamiltonian (28) it is easy to verify that for a general solution $|\xi_{-1}(t)|^2 - |\xi_1(t)|^2$ is not a constant of the motion. Thus, the constraint implied by (32) is not a general dynamical property of a solution. In order to verify that there are no other constraints than (30), so that $d = 2$, it suffices to show that if ζ^* is given, but arbitrary, one can solve the first three equations for γ^*, δ^*, and ϵ^*. The solvability condition is that

$$\begin{vmatrix} (\mu_1^*)^2 & (\mu_{-1})^2 & 0 \\ 0 & 2\mu_{-2}\mu_{-1}^* & 0 \\ 2\mu_{-2}^*\mu_1^* & 0 & 2\mu_1^*\mu_2 \end{vmatrix} = 4\mu_{-2}^*(\mu_1^*)^3\mu_2 \neq 0 \, . \tag{33}$$

Thus, except on the lower dimensional manifold $\mu_{-2}\mu_1\mu_2 = 0$, the system is solvable and $d = 2$, as claimed. Except that $d = 2$, one would expect an eight dimensional manifold of parameters for which the Hamiltonian system possesses solutions with explosive instabilities.

(3) An apparently overdetermined system, $\omega_{-3} = -3$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$,

$$W = 6\gamma\xi_{-3}\xi_1\xi_2 + 6\gamma^*\xi_{-3}^*\xi_1^*\xi_2^* + \rho_2\xi_2^*\xi_1^2 + \rho_2^*\xi_2(\xi_1^*)^2 + 2\rho_3\xi_3^*\xi_1\xi_2 + 2\rho_3^*\xi_3\xi_1^*\xi_{-2}^* \, , \tag{34}$$

for which the system (18) is

$$\begin{aligned} D_{-3}\mu_{-3}&=6\gamma^*\mu_1^*\mu_2^* , \\ D_1\mu_1&=6\gamma^*\mu_{-3}^*\mu_2^*+2\rho_2^*\mu_1^*\mu_2+2\rho_3^*\mu_2^*\mu_3 , \\ D_2\mu_2&=6\gamma^*\mu_{-3}^*\mu_1^*+\rho_2(\mu_1)^2+2\rho_3^*\mu_1^*\mu_3 , \\ D_3\mu_3&=+2\rho_3\mu_1\mu_2 . \end{aligned} \tag{35}$$

Since (35) consists of eight real equations in terms of 6 unknowns, d must be at least two. It is easy to obtain the complex consistency relation

$$D_1|\mu_1|^2-2D_2^*|\mu_2|^2+|\mu_3|^2(2D_3-D_3^*) +|\mu_{-3}|^2(2D_{-3}^*-D_{-3})=0 , \tag{36}$$

for which the imaginary part is, provided $\alpha\neq0$,

$$-3|\mu_{-3}|^2+|\mu_1|^2+2|\mu_2|^2+3|\mu_3|^2=0 \tag{37}$$

and the real part, after the use of (36),

$$-2|\mu_2|^2\beta=0 . \tag{38}$$

It is easy to conclude that provided $\mu_{-3}\mu_1\mu_2\mu_3\neq0$ and (36) holds, (35) possesses a unique solution. Thus, from (38) one infers that

$$\beta=0 \tag{39}$$

so that (37) and (39) are the solvability conditions for (35) and $d=2$.

It is known [6] that explosive instabilities exist for (34) if and only if

$$9|\gamma|^2>|\rho_3|^2 , \tag{40}$$

and it is interesting to see the result appear in this case. One finds easily from the first and last equations of (35)

$$\frac{3|\gamma|}{|\rho_3|}=\frac{|\mu_{-3}|^2}{|\mu_3|^2}=\frac{3|\mu_3|^2+2|\mu_2|^2+|\mu_1|^2}{3|\mu_3|^2}\geq 1 , \tag{41}$$

where (37) has been employed to eliminate $|\mu_{-3}|^2$. Although the domain of explosive instability Hamiltonians is six dimensional in the space of coefficients γ, ρ_2, ρ_3, it does not cover the full space, as (41) clearly shows.

B. Third order interaction of three waves

(1) An overdetermined system with a compact constant of the motion, $\omega_{-1}=-1$, $\omega_2=2$, $\omega_3=3$,

$$W=\gamma\xi_{-1}\xi_2^*\xi_3+\gamma^*\xi_{-1}^*\xi_2\xi_3^* , \tag{42}$$

for which (18) is

$$\begin{aligned} D_{-1}\mu_{-1}&=\gamma^*\mu_2\mu_3^* , \\ D_2\mu_2&=\gamma\mu_{-1}\mu_3 , \\ D_3\mu_3&=\gamma^*\mu_{-1}^*\mu_2 . \end{aligned} \tag{43}$$

Clearly the system (43) is greatly overdetermined and the consistency conditions for a solution are

$$D_{-1}|\mu_{-1}|^2=D_2^*|\mu_2|^2=D_3|\mu_3|^2 , \tag{44}$$

of which the imaginary part is

$$\alpha|\mu_{-1}|^2=-\alpha|\mu_2|^2=\alpha|\mu_3|^2 . \tag{45}$$

No nontrivial solution of (45), and thus (44), is possible for $\alpha\neq0$. Thus no explosive instabilities occur.

This example exhibits the possible effects of the existence of additional constants of the motion. For a Hamiltonian system (7) if ξ_α and ξ_β occur only in the combinations $\xi_\alpha\xi_\beta^*$ and $\xi_\alpha^*\xi_\beta$, so that

$$W=W(\xi_\alpha\xi_\beta^*,\xi_\alpha^*\xi_\beta,\xi_k,\xi_l^*) , \quad k\neq\alpha, l\neq\beta \tag{46}$$

it follows easily from (7) that

$$|\xi_\alpha|^2+|\xi_\beta|^2=\text{const} \tag{47}$$

and if

$$W=W(\xi_\alpha\xi_\beta,\xi_\alpha^*\xi_\beta^*,\xi_k,\xi_l^*) , \quad k\neq\alpha, l\neq\beta \tag{48}$$

then

$$|\xi_\alpha|^2-|\xi_\beta|^2=\text{const.} \tag{49}$$

Generalizations of (46)–(49) are readily made. For the Hamiltonian (42) it follows

$$|\xi_{-1}|^2+|\xi_2|^2=\text{const} \tag{50}$$

and

$$|\xi_2|^2+|\xi_3|^2=\text{const} \tag{51}$$

so that no explosive instability is possible.

(2) An overdetermined system with noncompact constants of the motion. If (42) is replaced by the Cherry-like system

$$W=\gamma\xi_1\xi_2\xi_{-3}+\gamma^*\xi_1^*\xi_2^*\xi_{-3}^* \tag{52}$$

then the system of linear equations is

$$\begin{aligned} D_{-3}\mu_{-3}&=\gamma^*\mu_1^*\mu_2^* , \\ D_1\mu_1&=\gamma^*\mu_{-3}^*\mu_2^* , \\ D_2\mu_2&=\gamma^*\mu_{-3}^*\mu_1^* , \end{aligned} \tag{53}$$

for which the consistency conditions are

$$D_{-3}|\mu_{-3}|^2=D_1|\mu_1|^2=D_2|\mu_2|^2 , \tag{54}$$

whose imaginary part yields ($\alpha\neq0$)

$$|\mu_{-3}|^2=|\mu_1|^2=|\mu_2|^2 \tag{55}$$

and whose real part requires $\beta=0$. Thus, explosively unstable solutions occur provided (55) holds. The dynamical system (52) possesses the constants of the motion, see (48) and (49),

$$|\xi_{-3}|^2-|\xi_1|^2=\text{const}, \tag{56}$$

$$|\xi_1|^2-|\xi_2|^2=\text{const} , \tag{57}$$

equivalent to (53) for explosive instability solutions. Thus, constants of the motion may or may not inhibit explosive instabilities. A more intricate example is given below in terms of a five wave resonant interaction. Slight generalizations of cases (1) and (2) are integrated explicitly in the Appendix.

(3) A case in which the interactions do not satisfy the resonance conditions (12) and (13). A variation of the interaction (42) is

$$W=\gamma\xi_{-1}\xi_2^*\xi_3+\gamma^*\xi_{-1}^*\xi_2\xi_3^* + \tfrac{1}{3}\{\delta\xi_2^3+\delta^*(\xi_2^*)^3+\epsilon\xi_3^3+\epsilon^*(\xi_3^*)^3\} . \quad (58)$$

The subscripts -1, 2, and 3 no longer refer to mode frequencies, but only indices that facilitate a comparison with (42). Consistent with the discussion of nonresonant Hamiltonians, one assumes $\beta=0$, but $\alpha\neq0$, so that the conditions for an explosive instability become

$$\begin{aligned} i\alpha\mu_{-1}&=\gamma^*\mu_2\mu_3^* ,\\ i\alpha\mu_2&=\gamma\mu_{-1}\mu_3+\delta^*(\mu_2^*)^2 ,\\ i\alpha\mu_3&=\gamma^*\mu_{-1}^*\mu_2+\epsilon^*(\mu_3^*)^2 . \end{aligned} \quad (59)$$

It is clear that the system is solvable for γ, δ, and ϵ for any nonzero values of α, μ_{-1}, μ_2, μ_3. The fact that no constraints of the form (20) exist is a direct verification that the system is not resonant. Since solutions exist without constraints, one expects the set of Hamiltonians with explosive instabilities to be of dimension six in the space defined by the real and imaginary parts of γ, δ, and ϵ.

For this simple problem a direct solution of (59) is possible, in which γ, δ, and ϵ are given and then α, μ_{-1}, μ_2, and μ_3 are determined. It is convenient to define ν_j by the relation

$$\mu_j=-i\alpha\nu_j , \quad (60)$$

so that

$$\begin{aligned} \nu_{-1}&=\gamma^*\nu_2\nu_3^* ,\\ \nu_2&=-\gamma\nu_{-1}\nu_3-\delta^*(\nu_2^*)^2 ,\\ \nu_3&=\gamma^*\nu_{-1}^*\nu_2-\epsilon^2(\nu_3^*)^2 . \end{aligned} \quad (61)$$

On elimination of ν_{-1} one finds

$$\begin{aligned} |\nu_2|^2&=\frac{-\delta^*(\nu_2^*)^2}{1+|\gamma|^2|\nu_3|^2} ,\\ |\nu_3|^2&=\frac{-\epsilon^*(\nu_3^*)^2}{1-|\gamma|^2|\nu_2|^2} , \end{aligned} \quad (62)$$

so that $\arg(\nu_2)$ and $\arg(\nu_3)$ are uniquely determined, and finally,

$$\left|1-\frac{|\gamma|^2}{|\delta|^2}\left[1+\frac{|\gamma|^2}{|\alpha|^2}|\mu_3|^2\right]^2\right|=\frac{|\epsilon|}{|\alpha|}|\mu_3| . \quad (63)$$

The relation (63) does not allow solutions for $|\mu_3|$ real and positive for all values of $|\gamma|,|\delta|$, and $|\epsilon|$. For instance, no real, positive roots exist for $|\gamma|>|\delta|$ and $|\epsilon|=0$, and thus by Rouché's lemma, also in some set with $|\epsilon|/|\alpha|$ sufficiently small. On the other hand, for $|\gamma|<|\delta|$ and $|\epsilon|/|\alpha|$ sufficiently small, real positive solutions clearly do exist. The dimensionality of the set of Hamiltonians with explosive instabilities is indeed six, although it is not the full space.

C. Third order interaction of five waves

$$\omega_{-2}=-2,\omega_{-1}=-1,\omega_1=1,\omega_2=2,\omega_3=3,$$

$$\begin{aligned} W=&\gamma\xi_{-1}\xi_2^*\xi_3+\gamma^*\xi_{-1}^*\xi_2\xi_3^*+\delta\xi_1^*\xi_2^*\xi_3+\delta^*\xi_1\xi_2\xi_3^*\\ &+\epsilon\xi_{-2}(\xi_{-1}^*)^2+\epsilon^*\xi_{-2}^*\xi_{-1}^2+\zeta\xi_{-2}\xi_1^2+\zeta^*\xi_{-2}^*(\xi_1^*)^2\\ &+\eta\xi_{-2}\xi_{-1}^*\xi_1+\eta^*\xi_{-2}^*\xi_{-1}\xi_1^* . \end{aligned} \quad (64)$$

The Hamiltonian (64) is a complicated extension of (42) involving five mode amplitudes ξ_{-2}, ξ_{-1}, ξ_1, ξ_2, ξ_3, and five complex constants γ, δ, ϵ, ζ, and η. Thus, it would seem reasonable that explosively growing modes would exist. However, ξ_2, ξ_2^*, ξ_3, and ξ_3^* occur only in the combinations $\xi_2\xi_3^*$ and $\xi_2^*\xi_3$, so that from (46) to (47) it follows for any solution

$$|\xi_2|^2+|\xi_3|^2=\text{const} . \quad (65)$$

Hence the only admissible solution with explosive instabilities would require

$$\xi_2=\xi_3=0 . \quad (66)$$

The interaction (64), with $\xi_2=\xi_3=0$, is quite similar to (28), and it is easy to verify that explosive instabilities occur. Thus, even with constants of the motion of the form (65), if there are enough other interacting modes, then explosive instabilities may easily occur.

D. Fourth order four wave interaction

This final example with $\omega_{-4}=-4$, $\omega_{-1}=-1$, $\omega_2=2$, $\omega_3=3$, and

$$\begin{aligned} W=&\gamma\xi_2\xi_3\xi_{-1}\xi_{-4}+\gamma^*\xi_2^*\xi_3^*\xi_{-1}^*\xi_{-4}^*\\ &+\delta\xi_2(\xi_{-1})^2+\delta^*\xi_2^*(\xi_{-1}^*)^2+\epsilon\xi_{-4}\xi_{-4}^*\xi_{-1}\xi_{-1}^* , \end{aligned} \quad (67)$$

exhibits another significant aspect of the problem. In the interaction Hamiltonian (67) ϵ must be real, so that W is characterized by five real interaction constants. The equations characterizing explosively growing modes may be written in the more symmetric form, derivable from (18) by multiplication by μ_j^*:

$$\begin{aligned} D_3|\mu_3|^2&=\gamma^*\mu_{-4}^*\mu_{-1}^*\mu_2^*\mu_3^* ,\\ D_2|\mu_2|^2&=\gamma^*\mu_{-4}^*\mu_{-1}^*\mu_2^*\mu_3^*+\delta^*\mu_2^*\mu_{-1}^2\mu_{-4}^* ,\\ D_{-1}|\mu_{-1}|^2&=\gamma^*\mu_{-4}^*\mu_{-1}^*\mu_2^*\mu_3^*\\ &\quad+2\delta\mu_2(\mu_{-1}^*)^2\mu_{-4}+\epsilon|\mu_{-4}|^2|\mu_{-1}|^2 ,\\ D_{-4}|\mu_{-4}|^2&=\gamma^*\mu_{-4}^*\mu_{-1}^*\mu_2^*\mu_3^*\\ &\quad+\delta^*\mu_2^*\mu_{-1}^2\mu_{-4}^*+\epsilon|\mu_{-4}|^2|\mu_{-1}|^2 . \end{aligned} \quad (68)$$

One can eliminate γ and δ in the third and fourth equations of (68) by the use of the first two relations and one finds

$$D_{-4}|\mu_{-4}|^2-D_2|\mu_2|^2=\epsilon|\mu_{-4}|^2|\mu_{-1}|^2 , \quad (69)$$

$$D_{-4}|\mu_{-4}|^2-D_{-1}|\mu_{-1}|^2+(2D_2^*-D_2)|\mu_2|^2 +(D_3-2D_3^*)|\mu_3|^2=0 , \quad (70)$$

and one may replace the last two equations of (68) with (69) and (70). Clearly the first two equations of (68) determine γ and δ provided none of μ_{-4}, μ_{-1}, μ_2, μ_3 vanish, so it suffices to study only (69) and (70). The imaginary part of (69) yields, provide $\alpha \neq 0$,

$$|\mu_{-4}|^2 = |\mu_2|^2 , \tag{71}$$

while the real part yields

$$\epsilon|\mu_{-4}|^2|\mu_{-1}|^2 = (4|\mu_{-4}|^2 + 2|\mu_2)^2\beta = 6\beta|\mu_{-4}|^2 \tag{72}$$

so that

$$\epsilon|\mu_{-1}|^2|\mu_{-4}|^2 = 6\beta|\mu_{-4}|^2 . \tag{73}$$

The imaginary part of (68) yields

$$0 = |\mu_{-4}|^2 - |\mu_{-1}|^2 - 3|\mu_2|^2 + 3|\mu_3|^2 . \tag{74}$$

The two relations (71) and (74) are equivalent to the resonance identity

$$0 = -4|\mu_{-4}|^2 - |\mu_{-1}|^2 + 2|\mu_2|^2 + 3|\mu_3|^2 . \tag{75}$$

Finally the real part of (70) gives

$$\beta[4|\mu_{-4}|^2 - |\mu_{-1}|^2 - 2|\mu_2|^2 - 3|\mu_3|^2] = 0 \tag{76}$$

or

$$-2\beta|\mu_{-1}|^2 = 0 . \tag{77}$$

If $\mu_{-1} = 0$, then there is no interaction and no explosive instability; equally if $\mu_{-4} = 0$ then $\mu_2 = 0$, and again no explosive instabilities exist; hence $\mu_{-1}\mu_{-4} \neq 0$ and from (77)

$$\beta = 0 , \tag{78}$$

while from (73)

$$\epsilon = 0 . \tag{79}$$

Provided (78) and (79) hold, then (69) and (70) reduce to (71) and (74), [or (75)], and explosive instabilities exist. In this case the dimensionality of the set of Hamiltonians with explosive instabilities is no more than 4, since $\epsilon = 0$, and the dimensionality is lower than the dimension of the set of Hamiltonians. Parenthetically, one can readily verify that

$$|\xi_2|^2 - |\xi_{-4}|^2 = \text{const} \tag{80}$$

for any solution of the dynamical system with interaction (67), although the property (80) does not in any way limit the possibility of explosive growth.

IV. SUMMARY

This paper examines the consequences of a multiple time scale expansion of a Hamiltonian system of M interacting modes. The expansion leads to the study of Hamiltonians homogeneous of degree N satisfying certain resonance conditions. An earlier work [6] established many general properties of such systems and showed that provided both positive- and negative-energy modes are present, explosive instabilities originating in similarity solutions may occur. This paper has addressed the same problem by an inverse procedure. A form of interaction Hamiltonian is given, corresponding to a homogeneous polynomial of degree N, which is consistent with the time-averaged resonant interaction hypothesis, and which is a linear function of the interaction constants. The question is then, for a given structure of explosive instability mode, is there a set of interaction constants which is consistent with that mode? Typically, provided there are enough interaction constants an explosive instability is possible for a domain of positive volume in the space of interaction constants. A positive volume also implies that a positive fraction of the entire space is associated with explosive instabilities. It is possible, however, that for special cases the domain of interaction constants with explosive instabilities is of lower dimension than the full space. The existence of additional constants of the motion may prevent explosive instabilities, or again it may not, even if the additional constant of the motion itself appears inconsistent with an explosive instability. All of these possibilities are shown in the many examples given. Thus, this work strongly supports the idea that explosive instabilities occur commonly for homogeneous interaction Hamiltonians, provided there are enough interaction terms between the various modes.

These results also have consequences for more general Hamiltonian system. In the neighborhood of a linearly stable equilibrium point, if there are modes of both positive- and negative-energy type, then explosive instabilities are likely for substantial ranges of values of the higher order resonant interaction constants. The many examples show that it is not possible to give absolute results, but provided many modes interact with many independent forces, the equilibrium point is most likely to be nonlinearly unstable for wide ranges of parameters.

Finally, we point out that possible symmetries, implying momentum conservation in certain directions, can lead to spatial resonance conditions which will not allow the required resonance conditions to be fulfilled exactly. According to results obtained in Ref. [5], such a mismatch of the frequencies should have the consequence that for nonlinear instabilities to occur nonzero minimum initial amplitudes would be required. Except for this modification, however, the general results should remain valid.

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APPENDIX

The observation that the Hamiltonians (42) and (52) have constants of the motion (50), (51), and (56), (57) as well as the Hamiltonian itself permits an explicit integration of the equations of motion in a somewhat simpler and more direct manner than employed in [4]. The Hamiltonian (42) may be extended somewhat to

$$W = \omega_{-1}|\xi_{-1}|^2 + \omega_2|\xi_2|^2 + \omega_3|\xi_3|^2 + \gamma\xi_{-1}\xi_2^*\xi_3 + \gamma^*\xi_{-1}^*\xi_2\xi_3^* , \tag{A1}$$

for which it is easily verified that

$$|\xi_{-1}|^2+|\xi_2|^2=\text{const}=k \tag{A2}$$

and

$$|\xi_2|^2+|\xi_3|^2=\text{const}=l\ . \tag{A3}$$

If one sets

$$\xi_2(t)=\rho(t)\epsilon^{i\theta_2(t)}\ , \tag{A4}$$

then from (A2) and (A3)

$$\xi_{-1}(t)=\sqrt{k-\rho^2(t)}\epsilon^{i\theta_{-1}(t)}\ , \tag{A5}$$

$$\xi_3(t)=\sqrt{l-\rho^2(t)}\epsilon^{i\theta_3(t)}\ , \tag{A6}$$

and from the constancy of W, one infers that there is a constant h' such that

$$\omega'\rho^2(t)+2|\gamma|\rho(t)\sqrt{[k-\rho^2(t)][l-\rho^2(t)]}\cos(\theta_{-1}-\theta_2+\theta_3+\delta)=h'\ , \tag{A7}$$

where

$$\gamma=|\gamma|e^{i\delta} \tag{A8}$$

and

$$\omega'=\omega_2-\omega_{-1}-\omega_3\ . \tag{A9}$$

The differential equation of motion for $\xi_2(t)$ is

$$i\xi_2^*\dot{\xi}_2=\omega_{+2}\xi_2\xi_2^*+\gamma\xi_{-1}\xi_2^*\xi_3\ , \tag{A10}$$

whose imaginary part, after the use of (A7) is

$$\rho(t)\dot{\rho}(t)=\{|\gamma|^2\rho^2(t)[k-\rho^2(t)][l-\rho^2(t)] -\tfrac{1}{2}[h'-\omega'\rho^2(t)]^2\}^{1/2}\ . \tag{A11}$$

Clearly $\rho(t)$ is determined by quadrature, and the phases $\theta_{-1}(t)$, $\theta_2(t)$, and $\theta_3(t)$ are subsequently also determined by quadrature. Thus the system given by (A1) is integrable.

For the Hamiltonian studied in [5], a generalization of Eq. (52),

$$W=\omega_{+1}|\xi_1|^2+\omega_2|\xi_2|^2+\omega_{-3}|\xi_{-3}|^2 +\gamma\xi_1\xi_2\xi_{-3}+\gamma^*\xi_1^*\xi_2^*\xi_{-3}^*\ , \tag{A12}$$

the two constants of the motion are

$$|\xi_1|^2-|\xi_2|^2=k\ , \tag{A13}$$

$$|\xi_{-3}|^2-|\xi_2|^2=l\ , \tag{A14}$$

so that with the definitions (A4), (A8), and

$$\omega'=\omega_1+\omega_2+\omega_{-3}\ , \tag{A15}$$

one finds, in analogy with (A11),

$$\rho(t)\dot{\rho}(t)=\{|\gamma|^2\rho^2(t)[k+\rho^2(t)][l+\rho^2(t)] -\tfrac{1}{2}[h'-\omega'\rho^2(t)]^2\}^{1/2}\ , \tag{A16}$$

a result given in [5]. The system with Hamiltonian (A12) is thus integrable.

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